9.3 (continued)

Boundary-Value Problems (BVP)

$$x'' + ax = f(t)$$

$\underbrace{x(0) = 0 \quad x(L) = 0}_{\text{values at the ends and not at initial } t}$ **Dirichlet condition (values specified)**

$x'(0) = 0, \; x'(L) = 0$ **Neumann condition (rates specified)**

let's look at $x'' + ax = 0 \qquad x(0) = x(L) = 0$

$$x(t) = A\cos(\sqrt{a}\,t) + B\sin(\sqrt{a}\,t)$$

$$x(0) = 0 \rightarrow A = 0$$

$$x(L) = 0 \rightarrow 0 = B\sin(\sqrt{a}\,L)$$

$B \neq 0$ (otherwise $x = 0$ for ALL t → trivial solution) then $\sin(\sqrt{a}L) = 0$ $\quad \sqrt{a}L = n\pi \quad n = 1, 2, 3, \dots$

for each n, the fundamental solution is

$X_n = \sin\left(\frac{n\pi}{L}t\right)$ (B is scaling constant, not important)

then general solution is a linear combination of ALL

$$\boxed{X(t) = b_1 \sin\left(\frac{\pi}{L}t\right) + b_2 \sin\left(\frac{2\pi}{L}t\right) + b_3 \sin\left(\frac{3\pi}{L}t\right) + \cdots}$$

this is solution to $X'' + aX = 0 \quad X(0) = X(L) = 0$

Sine series w/ half period L

if we had used the Neumann condition $X'(0) = X'(L) = 0$

we would get a cosine series.

now back to $\underbrace{x'' + ax}$ $= f(t)$ $\quad x(0) = x(L) = 0$

Sine series if $f(t) = 0$

$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$ each b_n is dependent on $f(t)$

we need to make $f(t)$ period (because solution x is periodic)

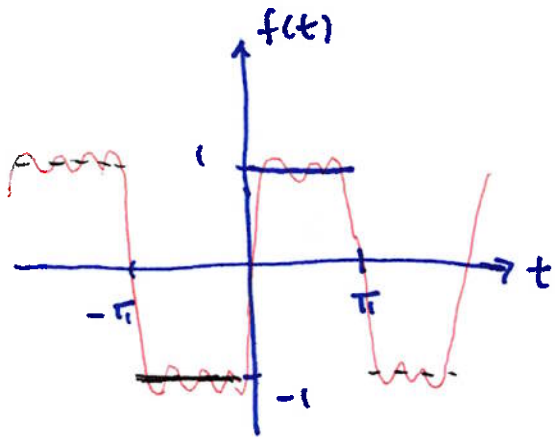
by adding either even or odd extensions

$\rightarrow$ pick the one that would make $f(t)$ satisfy the boundary conditions.

example $x'' + 2x = 1$ $\quad x(0)=0 \quad x(\pi)=0$

$\uparrow L = \pi$

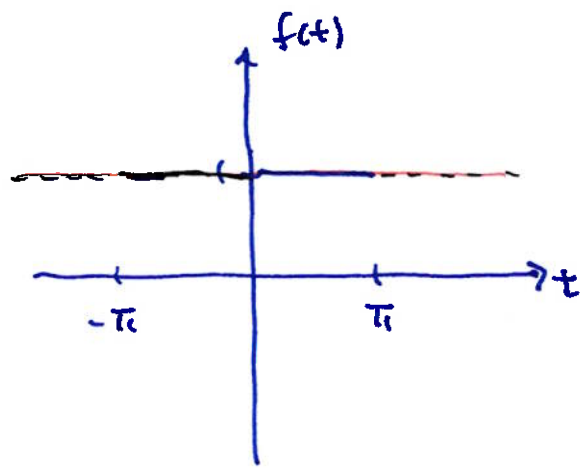
$f(t)=1$ let's look at both extensions to see which is appropriate (satisfies BC's)



w/ odd extensions

its Fourier series (in red)

notice it satisfies the BC's: $f(0)=f(\pi)=0$



w/ even extensions

its Fourier series (in red)

notice it does NOT satisfy BC's

BUT, notice it would satisfy

$x'(0) = x'(\pi) = 0$

for this example, we must pick odd extensions

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases} \qquad \text{period } 2\pi \quad (L = \pi)$$

its Fourier (sine) series is

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin(nt)$$

back to $\underbrace{x'' + 2x = 1}$ $\quad x(0) = x(\pi) = 0$

has sines series solution if $f(t) = 0$
w/ unknown coefficients

$$x(t) = \sum_{n=1}^{\infty} B_n \sin(nt)$$

Sub into the diff. eq. and replace
right side w/ its own series

$$x'(t) = \sum_{n=1}^{\infty} n B_n \cos(nt)$$

$$x''(t) = \sum_{n=1}^{\infty} -n^2 B_n \sin(nt)$$

$x'' + 2x = 1$ becomes

$$\sum_{n=1}^{\infty} -n^2 B_n \sin(nt) + 2\sum_{n=1}^{\infty} B_n \sin(nt) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin(nt)$$

for each $n = 1, 2, 3, \ldots$ equate coefficients of $\sin(nt)$

$$-n^2 B_n + 2B_n = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$B_n = \frac{2\left[1-(-1)^n\right]}{n\pi(2-n^2)} \qquad n = 1, 2, 3, \ldots$$

Solution: $$x(t) = \sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{n\pi(2-n^2)} \sin(nt)$$

$$= \frac{4}{\pi}\sin(t) - \frac{4}{21\pi}\sin(3t) - \frac{4}{105\pi}\sin(5t) - \ldots$$